Invariance of simultaneous similarity and equivalence of matrices under extension of the ground field

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Abstract

We give a new and elementary proof that simultaneous similarity and simultaneous equivalence of families of matrices are invariant under extension of the ground field, a result which is non-trivial for finite fields and first appeared in a paper of Klinger and Levy ([2]).

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1 Introduction

In this article, we let $\mathbb{K}$ denote a field, $\mathbb{L}$ a field extension of $\mathbb{K}$, and n and p two positive integers.

Definition 1. Two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ of matrices of $\mathrm{M}_n(\mathbb{K})$ indexed over the same set I are said to be **simultaneously similar** when there exists $P \in \mathrm{GL}_n(\mathbb{K})$ such that

$$\forall i \in I,\ P\,A_i\,P^{-1} = B_i$$

(such a matrix P will then be called a **base change matrix** with respect to the two families).

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Two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ of matrices of $\mathrm{M}_{n,p}(\mathbb{K})$ indexed over the same set I are said to be **simultaneously equivalent** when there exists a pair $(P,Q)\in \mathrm{GL}_n(\mathbb{K})\times \mathrm{GL}_p(\mathbb{K})$ such that

$$\forall i\in I,\ P\,A_i\,Q=B_i.$$

Of course, those relations extend the familiar relations of similarity and equivalence respectively on $\mathrm{M}_n(\mathbb{K})$ dans $\mathrm{M}_{n,p}(\mathbb{K})$, and they are equivalence relations respectively on $\mathrm{M}_n(\mathbb{K})^I$ dans $\mathrm{M}_{n,p}(\mathbb{K})^I$.
The simultaneous similarity of matrices is generally regarded upon as a "wild problem" where finding a useful characterisation by invariants seems out of reach. See [1] for an account of the problem and an algorithmic approach to its solution (for that last matter, also see [2]).

In this respect, our very limited goal here is to establish the following two results :

Theorem 1. *Let $\mathbb{K}-\mathbb{L}$ be a field extension and I be a set.*
Let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of matrices of $M_n(\mathbb{K})$.
Then $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously similar in $M_n(\mathbb{K})$ if and only if they are simultaneously similar in $M_n(\mathbb{L})$.

Theorem 2. *Let $\mathbb{K}-\mathbb{L}$ be a field extension and I be a set.*
Let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of matrices of $M_{n,p}(\mathbb{K})$.
Then $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously equivalent in $M_{n,p}(\mathbb{K})$ if and only if they are simultaneously equivalent in $M_{n,p}(\mathbb{L})$.

Remarks 1.

(i) In both theorems, the "only if" part is trivial.

(ii) It is an easy exercise to derive theorem 1 from theorem 2. However, we will do precisely the opposite !

2 A proof for simultaneous similarity

2.1 A reduction to special cases

In order to prove theorem 2, we will not, *contra* [2], try to give a canonical form for simultaneous similarity. Instead, we will focus on base change matrices and prove directly that if one exists in $\mathrm{M}_n(\mathbb{L})$, then another (possibly the same), also exists in $\mathrm{M}_n(\mathbb{K})$. To achieve this, we will prove the theorem in the two following special cases:

(i) $\mathbb{K}$ has at least n elements;

(ii) $\mathbb{K}-\mathbb{L}$ is a separable quadratic extension.

Assuming these cases have been solved, let us immediately prove the general case. Case (i) handles the situation where $\mathbb{K}$ is infinite. Assume now that $\mathbb{K}$ is finite, and choose a positive integer N such that $(\# \mathbb{K})^{2^N} \geq n$.
Since $\mathbb{K}$ is finite, there exists (see section V.4 of [3]) a tower of N quadratic separable extensions

$$\mathbb{K} \subset K_1 \subset K_2 \subset \cdots \subset K_N.$$

We let $\mathbb{M}$ denote a compositum extension of K_N and $\mathbb{L}$ (as extensions of $\mathbb{K}$) :

$$\begin{tikzcd}
\mathbb{K} \arrow[r, dash] \arrow[d, dash] & K_1 \arrow[r, dash] & K_2 \arrow[r, dash] & \cdots \arrow[r, dash] & K_N \arrow[d, dash] \\
\mathbb{L} \arrow[rrrr, dash] & & & & \mathbb{M}.
\end{tikzcd}$$

Assume the families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ of matrices of $\mathrm{M}_n(\mathbb{K})$ are simultaneously similar in $\mathrm{M}_n(\mathbb{L})$. Then they are also simultaneously similar in $\mathrm{M}_n(\mathbb{M})$. However, $\# K_N = (\# \mathbb{K})^{2^N} \geq n$, so this simultaneous similarity also holds in $\mathrm{M}_n(K_N)$. Using case (ii) by induction, when then obtain that that $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously similar in $\mathrm{M}_n(\mathbb{K})$.

2.2 The case $\# \mathbb{K} \geq n$

The line of reasoning here is folklore, but we reproduce the proof for sake of completeness. Let then $P \in \mathrm{GL}_n(\mathbb{L})$ be such that

$$\forall i \in I,\ P\, A_i\, P^{-1} = B_i,$$

so

$$\forall i \in I,\ P\, A_i = B_i\, P.$$

Let V denote the $\mathbb{K}$-vector subspace of $\mathbb{L}$ generated by the coefficients of P, and choose a basis $(x_1, \dots, x_N)$ of V. Decompose then

$$P = x_1\, P_1 + \cdots + x_N\, P_N$$

with $P_1, \dots, P_N$ in $\mathrm{M}_n(\mathbb{K})$, and let W be the $\mathbb{K}$-vector subspace of $\mathrm{M}_n(\mathbb{K})$ generated by the N-tuple $(P_1, \dots, P_N)$. Since the A_i's and the B_i's have all their coefficients in $\mathbb{K}$, the previous relations give :

$$\forall i \in I,\ \forall k \in [\![1, N]\!],\ P_k\, A_i = B_i\, P_k$$

hence

$$\forall i \in I,\ \forall Q \in W,\ Q\, A_i = B_i\, Q.$$

It thus suffices to prove that W contains a non-singular matrix.
However, the polynomial $\det(Y_1\,P_1+\cdots+Y_N\,P_N)\in\mathbb{K}[Y_1,\dots,Y_N]$ is homogeneous of total degree n and is not the zero polynomial because

$$\det(x_1.P_1+\cdots+x_N.P_N)=\det(P)\neq 0.$$

Since $n\leq \#\mathbb{K}$, we conclude that the map $Q\mapsto \det Q$ does not totally vanish on W, which proves that $W\cap \mathrm{GL}_n(\mathbb{K})$ is non-empty, QED.

2.3 The case Ł is a separable quadratic extension of $\mathbb{K}$

We choose an arbitrary element $\varepsilon\in Ł\smallsetminus\mathbb{K}$ and let σ denote the non-identity automorphism of the $\mathbb{K}$-algebra Ł. Assume $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously similar in $\mathrm{M}_n(Ł)$, and let $P\in\mathrm{GL}_n(Ł)$ be such that

$$\forall i\in I,\ P\,A_i\,P^{-1}=B_i.$$

We first point out that the problem is essentially unchanged should P be replaced with a $\mathbb{K}$-equivalent matrix of $\mathrm{GL}_n(Ł)$.
Indeed, let $(P_1,P_2)\in\mathrm{GL}_n(\mathbb{K})^2$, and set $P':=P_1\,P\,P_2^{-1}\in\mathrm{GL}_n(Ł)$, and $A'_i:=P_2\,A_i\,(P_2)^{-1}$ and $B'_i:=P_1\,B_i\,(P_1)^{-1}$ for all $i\in I$. Then :

$$\forall i\in I,\ P'\,A'_i\,(P')^{-1}=B'_i.$$

Since it follows directly from definition that $(A_i)_{i\in I}$ and $(A'_i)_{i\in I}$ are simultaneously similar in $\mathrm{M}_n(\mathbb{K})$, and that it is also true of $(B_i)_{i\in I}$ and $(B'_i)_{i\in I}$, it will suffice to show that $(A'_i)_{i\in I}$ and $(B'_i)_{i\in I}$ are simultaneously similar in $\mathrm{M}_n(\mathbb{K})$, knowing that they are simultaneously similar in $\mathrm{M}_n(Ł)$.

Returning to P, we split it as

$$P=Q+\varepsilon\,R\qquad\text{with } (Q,R)\in\mathrm{M}_n(\mathbb{K})^2.$$

The previous remark then reduces the proof to the case where the pair (Q,R) is canonical in terms of Kronecker reduction (see chapter XII of [4] and our section 4). More roughly, when can assume, since P is non-singular, that, for some $q\in[\![0,n]\!]$:

$$Q=\begin{bmatrix}M & 0\\ 0 & I_{n-q}\end{bmatrix}\quad\text{and}\quad R=\begin{bmatrix}I_q & 0\\ 0 & N\end{bmatrix}$$

where $M\in\mathrm{M}_q(\mathbb{K})$, N is a nilpotent matrix of $\mathrm{M}_{n-q}(\mathbb{K})$, and we have let I_k denote the unit matrix of $\mathrm{M}_k(\mathbb{K})$.

Let $i\in I$. Applying σ coefficient-wise to $P\,A_i\,P^{-1}=B_i$, we get:

$$\sigma(P)\,A_i\,\sigma(P)^{-1}=B_i=P\,A_i\,P^{-1},$$

hence A_i commutes with $\sigma(P)^{-1}\,P$. We now claim the following result:

Lemma 3. *Under the preceding assumptions, any matrix of $M_n(\mathbb{K})$ that commutes with $\sigma(P)^{-1}\,P$ also commutes with P.*

Assuming this lemma holds, we deduce that $\forall i \in I,\ P\,A_i\,P^{-1} = A_i$, hence $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are equal, thus simultaneously similar in $\mathrm{M}_n(\mathbb{K})$, which finishes our proof.

Proof of lemma 3. Let $A \in \mathrm{M}_n(\mathbb{K})$ which commutes with $\sigma(P)^{-1}\,P$. Applying σ, we deduce that A also commutes with $P^{-1}\sigma(P)$, hence with $I_n + (\sigma(\varepsilon) - \varepsilon)\,P^{-1}\,R$, hence with $P^{-1}\,R$ since $\sigma(\varepsilon) \neq \varepsilon$.
Notice then that

$$P^{-1}\,R = \begin{bmatrix} (M + \varepsilon.I_q)^{-1} & 0 \\ 0 & (I_{n-q} + \varepsilon\,N)^{-1}\,N \end{bmatrix}$$

with $(M + \varepsilon.I_q)^{-1}$ non-singular and $(I_n + \varepsilon\,N)^{-1}N$ nilpotent, so A, which stabilizes both $\mathrm{Im}(P^{-1}\,R)^n$ and $\mathrm{Ker}(P^{-1}\,R)^n$, must be of the form

$$A = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \quad \text{for some } (C, D) \in \mathrm{M}_q(\mathbb{K}) \times \mathrm{M}_{n-q}(\mathbb{K}).$$

Commutation of A with $P^{-1}\,R$ ensures that C commutes with $(M+\varepsilon.I_q)^{-1}$, whereas D commutes with $(I_{n-q}+\varepsilon\,N)^{-1}N = \varepsilon^{-1}.I_{n-q}-\varepsilon^{-1}.(I_{n-q}+\varepsilon\,N)^{-1}$ hence with $(I_{n-q} + \varepsilon\,N)^{-1}$. It follows that A commutes with P^{-1}, hence with P. $\square$

3 A proof for simultaneous equivalence

We will now derive theorem 2 from theorem 1. Under the assumptions of theorem 2, we choose an arbitrary object a that does not belong to I, and define

$$C_a = D_a := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{M}_{n+p}(\mathbb{K})$$

and, for $i \in I$,

$$C_i = \begin{bmatrix} 0 & A_i \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D_i = \begin{bmatrix} 0 & B_i \\ 0 & 0 \end{bmatrix} \quad \text{in } \mathrm{M}_{n+p}(\mathbb{K}).$$

The following two conditions are then equivalent :

(i) $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously equivalent ;

(ii) $(C_i)_{i\in I\cup\{a\}}$ and $(D_i)_{i\in I\cup\{a\}}$ are simultaneously similar.

Indeed, if condition (i) holds, then we choose $(P,Q) \in \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_p(\mathbb{K})$ such that $\forall i \in I,\ P\,A_i\,Q = B_i$, set $R := \begin{bmatrix} P & 0 \\ 0 & Q^{-1} \end{bmatrix}$, and remark that $R \in \mathrm{GL}_{n+p}(\mathbb{K})$ and

$$\forall i \in I \cup \{a\},\ R\,C_i\,R^{-1} = D_i.$$

Conversely, assume condition (ii) holds, and choose $R \in \mathrm{GL}_{n+p}(\mathbb{K})$ such that

$$\forall i \in I \cup \{a\},\ R\,C_i\,R^{-1} = D_i.$$

Equality $R\,C_a\,R^{-1} = C_a$ then entails that R is of the form

$$R = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \quad \text{for some } (P,Q) \in \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_p(\mathbb{K}),$$

and the other relations then imply that

$$\forall i \in I,\ P\,A_i\,Q^{-1} = B_i.$$

Using equivalence of (i) and (ii) with both fields $\mathbb{K}$ and Ł, theorem 2 follows easily from theorem 1.

4 Appendix : on the Kronecker reduction of matrix pencils

Attention was brought to me that, in [4], the proof that every pencil of matrix is equivalent to a canonical one fails for finite fields. We will give a correct proof here in the case of a “weak” canonical form (that is all we need here, and reducing further to a true canonical form is not hard from there using the theory of elementary divisors).

Notation 2. For $n \in \mathbb{N}$, set $L_n = \begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} \in \mathrm{M}_{n,n+1}(\mathbb{K})$ and $K_n = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix} \in \mathrm{M}_{n,n+1}(\mathbb{K})$; and, for arbitrary objects a and b, define the Jordan matrix:

$$J_n(a,b) = \begin{bmatrix} a & b & 0 & \\ 0 & a & b & \\ & & \ddots & \ddots \end{bmatrix} \in \mathrm{M}_n(\{0,a,b\}).$$

Theorem 4 (Kronecker reduction theorem for pencils of matrices). *Let A and B in $M_{n,p}(\mathbb{K})$. Then there are non-singular $(P_1, Q_1) \in GL_n(\mathbb{K}) \times GL_p(\mathbb{K})$ such that $P_1\,(A + X\,B)\,Q_1$ is block-diagonal with every non-zero diagonal block having one of the following forms, with only one of the first type:*

- *$P + X\,I_r$ for some non-singular $P \in GL_r(\mathbb{K})$;*
- *$J_r(1, X)$; $J_r(X, 1)$; $L_r + X K_r$; $(L_r + X K_r)^t$.*

This decomposition is unique up to permutation of blocks and up to similarity on the non-singular P.

We will only prove here that such a decomposition exists. Uniqueness is not needed here so we will leave it as an exercise for the reader.
We will consider A and B as linear maps from $E = \mathbb{K}^p$ to $F = \mathbb{K}^n$. Without loss of generality, we may assume $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$ and $\operatorname{Im} A + \operatorname{Im} B = F$. We define inductively two towers $(E_k)_{k\in\mathbb{N}}$ and $(F_k)_{k\in\mathbb{N}}$ of linear subspaces of E and F by:

(a) $E_0 = \{0\}$; $F_0 = A(\{0\}) = \{0\}$;

(b) $\forall k \in \mathbb{N},\ E_{k+1} = B^{-1}(F_k)$ and $F_{k+1} = A(E_{k+1})$.

Notice that $E_1 = \operatorname{Ker} B$. The sequences $(E_k)_{n\geq 0}$ and $(F_k)_{n\geq 0}$ are clearly non-decreasing so we can find a smallest integer N such that $E_N = E_k$ for every $k \geq N$. Hence $F_N = F_k$ for every $k \geq N$, and $E_N = g^{-1}(F_N)$. It follows that $A(E_N) = F_N$ and $B(E_N) \subset F_N$. We now let f and g denote the linear maps from E_N to F_N induced by A and B.
From there, the proof has two independent major steps:

Lemma 5. *There are basis* **B** *and* **C** *respectively of E_N and F_N such that $M_{\mathbf{B},\mathbf{C}}(f) + X\,M_{\mathbf{B},\mathbf{C}}(g)$ is block-diagonal with all non-zero blocks having one of the forms $J_r(1, X)$ or $L_s + X\,K_s$.*

Lemma 6. *There are splittings $E = E_N \oplus E'$ and $F = F_N \oplus F'$ such that $A(E') \subset F'$ and $B(E') \subset F'$.*

Assuming those lemmas are proven, let us see how we can easily conclude:

- We deduce from the two previous lemmas that $A + X\,B$ is $\mathbb{K}$-equivalent to some $\begin{bmatrix} A' + X\,B' & 0 \\ 0 & C(X) \end{bmatrix}$ where $C(X)$ is block-diagonal with all non-zero blocks of the form $J_r(1, X)$ or $L_s + X\,K_s$, and A' and B' have coefficients in $\mathbb{K}$, with $\operatorname{Ker} B' = \{0\}$; it will thus suffice to prove the existence of a canonical form for the pair (A', B');
- applying the first step of the proof to the matrices $(A')^t$ and $(B')^t$, we find that $A' + X\,B'$ is $\mathbb{K}$-equivalent to some $\begin{bmatrix} A'' + X\,B'' & 0 \\ 0 & D(X) \end{bmatrix}$

where $D(X)$ is block-diagonal with all non-zero blocks of the form $J_r(1,X)^t$ (which is $\mathbb{K}$-similar to $J_r(1,X)$) or $(L_s + X\,K_s)^t$, and A'' and B'' have coefficients in $\mathbb{K}$, with $\operatorname{Ker} B'' = \{0\}$ and $\operatorname{coker} B'' = \{0\}$. It follows that B'' is non-singular.

- Finally, $(B'')^{-1}(A'' + X\,B'') = (B'')^{-1}A'' + X.I_k$ for some integer k, and the pair (A'', B'') can thus be reduced by using the Fitting decomposition of $(B'')^{-1}A''$ combined with a Jordan reduction of its nilpotent part: this yields a block-diagonal matrix $\mathbb{K}$-equivalent to $A'' + X\,B''$ with all diagonal blocks of the form $J_r(X,1)$ or $P + X.I_s$ for some non-singular P. This completes the proof of existence.

Proof of lemma 6. We proceed by induction.
Assume, for some $k \in [\![1,N]\!]$, that there are splittings $E = E_N \oplus E'$ and $F = F_N \oplus F'$ such that $A(E') \subset F' \oplus F_k$ and $B(E') \subset F' \oplus F_k$. Since $B^{-1}(F_N) = E_N$, the subspaces F_N and $B(E')$ are independant. We can therefore find some F'' such that $F' \oplus F_k = F'' \oplus F_k$, $F_N \oplus F'' = F$ and $B(E') \subset F''$. Choose then a basis $(e_1, \dots, e_p)$ of E', and decompose $A(e_i) = f_i + f_i'$ for all $i \in [\![1,p]\!]$, with $f_i \in F''$ and $f_i' \in F_k$. For $i \in [\![1,p]\!]$, we have $f_i' = A(g_i)$ for some $g_i \in E_k$. Then $(e_1 - g_1, \dots, e_p - g_p)$ still generates a supplementary subspace E'' of E_N in E, and we now have $A(e_i - g_i) \in F''$ and $B(e_i - g_i) \in F'' \oplus F_{k-1}$ for all $i \in [\![1,p]\!]$. Hence $E = E_N \oplus E''$ and $F = F_N \oplus F''$, now with $A(E'') \subset F'' \oplus F_{k-1}$ and $B(E'') \subset F'' \oplus F_{k-1}$. The condition is thus proven at the integer $k-1$. By downward induction, we find that it holds for $k = 0$, QED. □

Proof of lemma 5. The argument is similar to the standard proof of the Jordan reduction theorem.

- Split $F_N = F_{N-1} \oplus W_{N,N}$ and $E_N = E_{N-1} \oplus V_{N,N} \oplus V'_{N,N}$ such that $E_{N-1} \oplus V'_{N,N} = E_{N-1} + (E_N \cap \operatorname{Ker} f)$, $V'_{N,N} \subset \operatorname{Ker} f$ and $f(V_{N,N}) = W_{N,N}$ (so f induces an isomorphism from $V_{N,N}$ to $W_{N,N}$). Set $W_{N,N-1} = g(V_{N,N})$ and $W'_{N,N-1} = g(V'_{N,N})$. Remark that $F_{N-2} \oplus W_{N,N-1} \oplus W'_{N,N-1} \subset F_{N-1}$, and split $F_{N-1} = F_{N-2} \oplus W_{N,N-1} \oplus W'_{N,N-1} \oplus W_{N-1,N-1}$.
- We then proceed by downward induction to define four families of linear subspaces $(V_{\ell,k})_{1 \le k \le \ell \le N}$, $(V'_{\ell,k})_{1 \le k \le \ell \le N}$ $(W_{\ell,k})_{1 \le k \le \ell \le N}$ and $(W'_{\ell,k})_{1 \le k \le \ell - 1 \le N-1}$ such that:
  (i) for every $k \in [\![1,N]\!]$,
  $$E_k = E_{k-1} \oplus V_{k,k} \oplus V_{k+1,k} \oplus \cdots \oplus V_{N,k} \oplus V'_{k,k} \oplus V'_{k+1,k} \oplus \cdots \oplus V'_{N,k};$$
  (ii) for every $k \in [\![1,N]\!]$,
  $$F_k = F_{k-1} \oplus W_{k,k} \oplus W_{k+1,k} \oplus \cdots \oplus W_{N,k} \oplus W'_{k+1,k} \oplus W'_{k+2,k} \oplus \cdots \oplus W'_{N,k};$$

(iii) for every $k \in [\![1,N]\!]$, $E_{k-1} + (E_k \cap \operatorname{Ker} f) = E_{k-1} \oplus V'_{k,k}$ and $V'_{k,k} \subset \operatorname{Ker} f$;

(iv) for every $\ell \in [\![1,N]\!]$ and $k \in [\![2,\ell]\!]$, g induces an isomorphism $g_{\ell,k} : V_{\ell,k} \xrightarrow{\simeq} W_{\ell,k-1}$ and an isomorphism $g'_{\ell,k} : V'_{\ell,k} \xrightarrow{\simeq} W'_{\ell,k-1}$;

(v) for every $\ell \in [\![1,N]\!]$ and $k \in [\![1,\ell]\!]$, f induces an isomorphism $f_{\ell,k} : V_{\ell,k} \xrightarrow{\simeq} W_{\ell,k}$ and, if $k < \ell$, an isomorphism $f'_{\ell,k} : V'_{\ell,k} \xrightarrow{\simeq} W'_{\ell,k}$.

$$\begin{tikzcd}
 & V_{\ell,1} \arrow[dl, "g"'] \arrow[d, "f", "\simeq"'] & \arrow[dl, "g"'] & \cdots & V_{\ell,\ell-1} \arrow[dl, "g"', "\simeq"] \arrow[d, "f", "\simeq"'] & V_{\ell,\ell} \arrow[dl, "g"', "\simeq"] \arrow[d, "f", "\simeq"'] \\
\{0\} & W_{\ell,1} & \cdots & & W_{\ell,\ell-1} & W_{\ell,\ell}.
\end{tikzcd}$$

$$\begin{tikzcd}
 & V'_{\ell,1} \arrow[dl, "g"'] \arrow[d, "f", "\simeq"'] & \arrow[dl, "g"'] & \cdots & V'_{\ell,\ell-1} \arrow[dl, "g"', "\simeq"] \arrow[d, "f", "\simeq"'] & V'_{\ell,\ell} \arrow[dl, "g"', "\simeq"] \arrow[d, "f"] \\
\{0\} & W'_{\ell,1} & \cdots & & W'_{\ell,\ell-1} & \{0\}.
\end{tikzcd}$$

- Set $\ell \in [\![1,N]\!]$. Define

$$G_\ell = V_{\ell,1} \oplus \cdots \oplus V_{\ell,\ell}, \quad G'_\ell = V'_{\ell,1} \oplus \cdots \oplus V'_{\ell,\ell},$$

$$H_\ell = W_{\ell,1} \oplus \cdots \oplus W_{\ell,\ell} \quad \text{and} \quad H'_\ell = W'_{\ell,1} \oplus \cdots \oplus W'_{\ell,\ell-1}.$$

  Notice that:

$$f(G_\ell) = H_\ell, \quad g(G_\ell) \oplus W_{\ell,\ell} = H_\ell, \quad f(G'_\ell) = H'_\ell \quad \text{and} \quad g(G'_\ell) = H'_\ell.$$

  From there, it is easy to conclude.

- Let $n_\ell = \dim W_{\ell,\ell}$. Remark that $\dim V_{\ell,k} = \dim W_{\ell,k} = n_\ell$ for every $1 \in [\![1,\ell]\!]$ and choose a basis $\mathbf{C}_{\ell,\ell}$ of $W_{\ell,\ell}$. Define $\mathbf{B}_{\ell,\ell} = f_{\ell,\ell}^{-1}(\mathbf{C}_{\ell,\ell})$, $\mathbf{C}_{\ell,\ell-1} := g_{\ell,\ell}(\mathbf{B}_{\ell,\ell})$ and proceed by induction to recover a basis for $V_{\ell,k}$ and $W_{\ell,k}$ for every suitable k: by glueing together those basis, we recover respective basis $(\mathbf{B}_{\ell,1}, \dots, \mathbf{B}_{\ell,\ell})$ and $(\mathbf{C}_{\ell,1}, \dots, \mathbf{C}_{\ell,\ell})$ of G_ℓ and H_ℓ and remark that f and g induce linear maps from G_ℓ to H_ℓ with respective matrices $L_\ell \otimes I_{n_\ell}$ and $K_\ell \otimes I_{n_\ell}$ in those basis (remember that $E_1 = \operatorname{Ker} g$). A simple permutation of basis shows that those linear maps can be represented by $I_{n_\ell} \otimes L_\ell$ and $I_{n_\ell} \otimes K_\ell$ in a suitable common pair of basis.

- Proceeding similarly for G'_ℓ and H'_ℓ, but starting from a basis of $V'_{\ell,\ell}$, we obtain that f and g induce linear maps from G'_ℓ to H'_ℓ and there is a suitable choice of basis so that their matrices are respectively $I_s \otimes I_\ell$ and $I_s \otimes J_\ell(0,1)$ for some integer s.
- Notice that we have defined splittings

$$E_N = G_1 \oplus G'_1 \oplus G_2 \oplus G'_2 \oplus \cdots \oplus G_N \oplus G'_N$$

and

$$F_N = H_1 \oplus H'_1 \oplus H_2 \oplus H'_2 \oplus \cdots \oplus H'_{N-1} \oplus H_N,$$

therefore lemma 5 is proven by glueing together the various basis built here.

□

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